

DUNKL OPERATORS AS COVARIANT DERIVATIVES IN A QUANTUM PRINCIPAL BUNDLE

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ABSTRACT. A quantum principal bundle is constructed for every Coxeter group acting on a finite-dimensional Euclidean space E , and then a connection is also defined on this bundle. The covariant derivatives associated to this connection are the Dunkl operators, originally introduced as part of a program to generalize harmonic analysis in Euclidean spaces. This gives us a new, geometric way of viewing the Dunkl operators. In particular, we present a new proof of the commutativity of these operators among themselves as a consequence of a geometric property, namely, that the connection has curvature zero.

1. INTRODUCTION

A development in modern harmonic analysis is the generalization of the partial derivative operators acting on functions on Euclidean space to the larger class of Dunkl operators. This theory was introduced in 1989 by Dunkl in his paper [4]. See [13], [6], [16], [17] and [18] as well as references therein for further mathematical developments, including generalizations of the Laplacian operator (known as the Dunkl Laplacian) and the Fourier transform (known as the Dunkl transform). This theory has had applications as well in mathematical physics and probability theory; these include studies of Calogero-Sutherland models, generalized Segal-Bargmann spaces and their associated integral kernel transforms as well as Markov processes generalizing Brownian motion. See [16], [17], [18] and [21] and references therein for more details.

Also in the 1980's there arose interest in a new type of geometry, which has usually been called noncommutative geometry (see [2]) but also called quantum geometry (see [15] for a different formalization of similar geometrical ideas), and along similar lines a new type of group theory known as quantum group theory (see [3] and [22]). Following the approach of Woronowicz (see [22] and [23]), the first author has developed an extensive theory of quantum principal bundles. (See [7], [8], [9] and [10].) This theory includes connections on such bundles as well as their associated covariant derivatives and curvature. In this paper this latter theory is applied to construct a quantum principal bundle and a connection on it with the property that the associated covariant derivatives are exactly the Dunkl operators. Using this identification we provide a new, geometric proof that the Dunkl operators commute among themselves. This proof is based on the fact, which we prove, that the curvature of the connection is zero.

The covariant derivatives of a connection in classical differential geometry are local operators. Therefore Dunkl operators, being non-local when the multiplicity function is non-zero, can not be covariant derivatives in the classical context. So our results show the power of using non-commutative structures. A way of describing

our results in physics terminology is that the theory of Dunkl operators is a type of non-commutative gauge theory.

The paper is organized as follows. In the next section we review the definitions and basic facts about root systems, Coxeter groups and Dunkl operators. We then have a section with the results of this paper and their proofs. In two appendices we present some basic material about quantum principal bundles and the differential calculi for finite classical groups.

2. DUNKL OPERATORS

We now review some basic properties of Dunkl operators, which were introduced in [4]. Other references for this material are [11], [12] and [17]. These may be consulted for more details and proofs.

We let \mathbb{R}^n for $n \geq 1$ denote the finite dimensional Euclidean space equipped with the standard inner product $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. For any $x \in \mathbb{R}^n$ we use its standard Euclidean norm $\|x\| = \sqrt{\langle x, x \rangle}$.

We would like to emphasize the intrinsic geometrical nature of our constructions. Accordingly we shall be working within a given finite-dimensional Euclidean vector space E , providing a framework for our considerations which are independent of a particular choice of coordinates. Being a finite-dimensional Euclidean space E is always isomorphic to one of the standard spaces \mathbb{R}^n , where $n = \dim(E)$. These isomorphisms are in a natural correspondence with the orthonormal frames in E .

For any $0 \neq \alpha \in E$ we denote by σ_α the orthogonal reflection in the hyperplane H_α orthogonal to α , namely $H_\alpha = \{x \in E \mid \langle \alpha, x \rangle = 0\}$. Then we have the explicit formula

$$(x)\sigma_\alpha \equiv x\sigma_\alpha = x - \frac{2\langle \alpha, x \rangle}{\|\alpha\|^2} \alpha$$

for all $x \in E$. A simple calculation shows that $\langle x\sigma_\alpha, y\sigma_\alpha \rangle = \langle x, y \rangle$ holds for all $x, y \in E$, that is, $\sigma_\alpha \in O(E)$, the orthogonal group of E . One can easily prove many elementary properties such as $\det(\sigma_\alpha) = -1$ and $\sigma_\alpha^2 = I$, the identity element in $O(E)$.

The reason for writing σ_α on the right is that in geometry the action of a group on the total space of a principal bundle is by convention a right action.

Definition 2.1. A *root system* is a finite set R of non-zero vectors in E which satisfies the following properties:

- (1) $\alpha \in R \Rightarrow -\alpha \in R$,
- (2) $\alpha \in R$ and $r\alpha \in R$ for some $r \in \mathbb{R} \Rightarrow r = \pm 1$,
- (3) $(R)\sigma_\alpha = R$ for all $\alpha \in R$.
- (4) $\|\alpha\|^2 = 2$ for each root $\alpha \in R$.

Remark 2.2. Even though Property 3 implies Property 1 since $\alpha\sigma_\alpha = -\alpha$, redundancy does no harm. Property 4 is a standard normalization convention. Since the roots α are only used to define the reflections σ_α and $\sigma_{r\alpha} = \sigma_\alpha$ for any $0 \neq r \in \mathbb{R}$, this normalization does not effect our results while it allows for compatibility with some other authors.

Given such a root system R , we denote the subgroup of $O(E)$ generated by the elements σ_α for all $\alpha \in R$ as $G \equiv G(R)$. (It turns out that this group is finite, since it is isomorphic to a subgroup of the group of permutations of R , which itself is a finite group. See [12] for a proof.) We say that G is the (*finite*) *Coxeter group* associated with the root system R .

A G -invariant function $\varkappa : R \rightarrow \mathbb{C}$ is called a *multiplicity function*. (Note that we are using Property 3 above of a root system here, since we are requiring that $\varkappa_{\alpha g} = \varkappa_\alpha$ for $g \in G$ and $\alpha \in R$ and so we need to know that $\alpha g \in R$, the domain of \varkappa .)

For a fixed root system R the set of all multiplicity functions defined on R forms a finite dimensional vector space over \mathbb{C} of dimension equal to the number of G -orbits in R .

We note that according to our definition $R = \emptyset$, the empty set, is a root system whose associated group consists of exactly one element (the identity), and therefore the trivial subgroup of $O(E)$ is a Coxeter group. Non-trivial examples of root systems and their associated Coxeter groups are given in [17]. See the text [12] for much more information on Coxeter groups.

We let R^+ denote the subset of positive elements in R with respect to a given total order on E . (An *order* on E is a partial order $<$ such that $u < v \Rightarrow u + w < v + w$ and $ru < rv$ for all $u, v, w \in E$ and $r > 0$. Such an order is said to be *total* if for all $u, v \in E$ either $u < v$ or $v < u$ or $u = v$. We say $u \in E$ is *positive* if $0 < u$.) We define the Dunkl operators below in terms of the subset R^+ of positive elements with respect to a given total order (which *do* exist), since this is how it is usually done in the literature. However, this does not depend on the particular choice of total order. This is due to various basic facts the reader can verify such as $R = R^+ \cup (-R^+)$ (a disjoint union), $\mathbb{R}\alpha \cap R = \{\alpha, -\alpha\}$ for all $\alpha \in R$, $\sigma_{-\alpha} = \sigma_\alpha$, $\alpha\sigma_\alpha = -\alpha$ and $\varkappa_{-\alpha} = \varkappa_\alpha$.

Definition 2.3. For any $\xi \in E$ and multiplicity function $\varkappa : R \rightarrow \mathbb{C}$ we define the *Dunkl operator* $T_{\xi, \varkappa}$ by

$$T_{\xi, \varkappa} f(x) := \partial_\xi f(x) + \sum_{\alpha \in R^+} \varkappa_\alpha \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} (f(x) - f(x\sigma_\alpha)),$$

where $\partial_\xi = \langle \xi, \text{grad} \rangle$ is the directional derivative associated to $\xi \in E$ (with $\text{grad} = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ being the usual gradient operator in orthogonal coordinates), $x \in E$ and $f \in C^1(E)$ is a complex valued function. This definition can equivalently be written as

$$T_{\xi, \varkappa} f(x) = \partial_\xi f(x) + \frac{1}{2} \sum_{\alpha \in R} \varkappa_\alpha \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} (f(x) - f(x\sigma_\alpha)),$$

which shows that this operator does not depend on the choice of the total order.

Note that the linear operator $T_{\xi, \varkappa}$ depends linearly on ξ . For the constant multiplicity function $\varkappa \equiv 0$ or for $R = \emptyset$ (either case being called the *trivial Dunkl structure*) the operator $T_{\xi, \varkappa}$ reduces to the directional derivative ∂_ξ associated to the vector $\xi \in E$.

Of course this definition of $T_{\xi, \kappa} f(x)$ only makes sense for $x \notin \cup_{\alpha} H_{\alpha}$, where H_{α} is the hyperplane defined above. However, we have the identity

$$\frac{f(x) - f(x\sigma_{\alpha})}{\langle \alpha, x \rangle} = \int_0^1 dt \partial_{\alpha} f(tx + (1-t)x\sigma_{\alpha})$$

for all $x \notin H_{\alpha}$. This allows one to use the expression on the right side to define the Dunkl operator at every $x \in E$, provided that f is a C^1 function so that the directional derivative $\partial_{\alpha} f$ is defined and continuous. Specifically for $x \in H_{\alpha}$ (that is, $x\sigma_{\alpha} = x$) the right side of the previous equation becomes $\partial_{\alpha} f(x)$. The Dunkl operators also have realizations in other function spaces, most notably as anti-Hermitian operators acting in $L^2(E, w_{\kappa} d^n x)$, where $d^n x$ denotes Lebesgue measure on E and w_{κ} is a non-negative function defined on E .

For us one important result is the generalization of the Leibniz rule to this context. Since this result is not emphasized in the literature, we now present the statement and proof.

Theorem 2.4. *For any $f, g : E \rightarrow \mathbb{C}$ which are C^1 and for any $x \in E$ we have the following generalization of the Leibniz formula:*

$$\begin{aligned} (T_{\xi, \kappa}(fg))(x) &= (T_{\xi, \kappa} f)(x)g(x) + f(x)(T_{\xi, \kappa} g)(x) \\ &\quad - \sum_{\alpha \in R^+} \kappa_{\alpha} \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} (f(x) - f(x\sigma_{\alpha}))(g(x) - g(x\sigma_{\alpha})) \end{aligned}$$

Proof. We prove the result for all $x \notin \cup_{\alpha} H_{\alpha}$ and use the previous comments to extend it to all $x \in E$. We use the fact that directional derivatives satisfy the usual Leibniz rule and proceed as follows:

$$\begin{aligned} (T_{\xi, \kappa}(fg))(x) &- (T_{\xi, \kappa} f)(x)g(x) - f(x)(T_{\xi, \kappa} g)(x) \\ &= \sum_{\alpha \in R^+} \kappa_{\alpha} \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} \left(f(x)g(x) - f(x\sigma_{\alpha})g(x\sigma_{\alpha}) \right. \\ &\quad \left. - (f(x) - f(x\sigma_{\alpha}))g(x) - f(x)(g(x) - g(x\sigma_{\alpha})) \right) \\ &= \sum_{\alpha \in R^+} \kappa_{\alpha} \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} \left(-f(x)g(x) + f(x\sigma_{\alpha})g(x) + f(x)g(x\sigma_{\alpha}) - f(x\sigma_{\alpha})g(x\sigma_{\alpha}) \right) \\ &= - \sum_{\alpha \in R^+} \kappa_{\alpha} \frac{\langle \alpha, \xi \rangle}{\langle \alpha, x \rangle} (f(x) - f(x\sigma_{\alpha}))(g(x) - g(x\sigma_{\alpha})) \end{aligned}$$

and the theorem is proved. \square

We have an immediate consequence of this theorem.

Corollary 2.5. *Suppose that f and g are as in the previous theorem. Suppose that at least one of these functions is G -invariant. Then we have the usual Leibniz formula*

$$(T_{\xi, \kappa}(fg))(x) = (T_{\xi, \kappa} f)(x)g(x) + f(x)(T_{\xi, \kappa} g)(x)$$

for all $x \in E$.

A non-trivial result of this theory is that the operators $T_{\xi, \kappa}$ and $T_{\eta, \kappa}$ commute for all $\xi, \eta \in E$. This fundamental result has several proofs in the literature, starting with the proof in Dunkl's seminal article [4]. We will offer a new proof later on.

3. RESULTS

We shall now apply a general formalism of connections and quantum principal bundles to a special case when the structure group G is a finite Coxeter group associated to a root system R in a finite-dimensional Euclidean vector space E . We shall derive some of the basic expressions and properties for Dunkl operators as consequences of geometrical conditions involving connections and their covariant derivatives and curvature. The reader should consult the appendices for all new notation. We also use the elegant Sweedler's notation for coproducts and co-actions, thereby avoiding an excessive amount of summations and indices.

Being a Coxeter group, G possesses a distinguished differential calculus Γ . It is based on the set S of all reflections of G . A property of Coxeter groups is that every reflection $s \in G$ has the form $s = \sigma_\alpha$ where $\alpha \in R$ is a root vector. (See [12].) Since the conjugation by any element of G of a reflection is again a reflection and since the reflections are involutive, the associated calculus Γ turns out to be bicovariant and $*$ -covariant. The elements $\pi(s) = [s]$ where $s \in S$ form a canonical basis in the complex vector space of left-invariant elements Γ_{inv} .

We define the total space of a principal bundle by

$$P := E \setminus (\cup_{\alpha \in R} H_\alpha),$$

which is an open, dense subset of E . The natural right action of G on E (given in the previous section) leaves P invariant. Moreover, the restricted action of G on P is free, since P consists exactly of all the vectors in E having trivial stabilizers in G . In this way we have defined a classical principal bundle $P \rightarrow P/G$. The space P/G is diffeomorphic to any of the connected components of P . It is called an open Weyl chamber. So P is equipped with the classical differential calculus, based on smooth differential forms. However, these forms will be interpreted as *horizontal forms* on P . The full differential calculus $\Omega(P)$ will be quantum and constructed as explained in Appendix A. We wish to emphasize that even though the spaces $(G, P$ and $P/G)$ of this principal bundle are all classical, the differential calculus which we will use is quantum and not classical.

By definition of the Coxeter group associated to a root system, we have

$$(3.1) \quad g^{-1} \sigma_\alpha g = \sigma_{\alpha g},$$

where we always use the convention that G acts on the right in E .

Let ϖ be the canonical flat connection, as defined in Appendix A. Arbitrary connections ω are then given by connection displacement maps $\lambda: \Gamma_{inv} \rightarrow \mathfrak{hor}^1(P)$ so that $\omega = \varpi + \lambda$. Such maps λ constitute the vector space associated to the affine space of all connections on P , and satisfy two characterizing conditions: hermicity and covariance. We shall now analyze possible forms for the connection displacement maps λ .

If we define

$$(3.2) \quad \lambda[\sigma_\alpha](x) = i h_\alpha(x) \alpha,$$

where h_α are smooth real functions on P satisfying

$$(3.3) \quad h_{\alpha g}(x) = h_\alpha(x g^{-1}),$$

$$(3.4) \quad h_{-\alpha}(x) = h_\alpha(x),$$

then such a map will be a connection displacement. Here the root vectors α are interpreted in a natural way as one-forms on P .

Remark 3.1. The presence of the imaginary unit $i = \sqrt{-1}$ in the above formula is due to the fact that we are considering real connections which intertwine the corresponding $*$ -structures. It is worth mentioning that $[\sigma_\alpha]^* = -[\sigma_\alpha]$, that is, the generators of our canonical differential calculus are imaginary.

The first condition ensures the covariance of λ while the second condition is a necessary consistency requirement, because the value of λ on $[\sigma_\alpha]$ obviously should not change if we replace α by $-\alpha$. (Recall that $\sigma_{-\alpha} = \sigma_\alpha$.) Let us explicitly verify the covariance property. It reads

$$ih_\alpha(xg)\alpha g^{-1} = (\lambda[\sigma_\alpha])_g(x) = \lambda[g\sigma_\alpha g^{-1}](x) = \lambda[\sigma_{\alpha g^{-1}}](x) = ih_{\alpha g^{-1}}(x)\alpha g^{-1},$$

and this is equivalent to the covariance condition for the functions h_α . We require these functions to be real, so that the resulting connection ω is real.

Definition 3.2. Connections $\omega = \varpi + \lambda$, where λ is given by (3.2), will be called *Dunkl connections*.

Let us also observe that using the same type of functions h_α we can generate another class of ‘spherically symmetrical’ displacements where

$$(3.5) \quad \lambda[\sigma_\alpha](x) = ih_\alpha(x)\zeta(x)$$

where ζ is the canonical radial one-form given by

$$(3.6) \quad \zeta(x) = \frac{1}{2}D\langle x, x \rangle.$$

Proposition 3.3. *The covariant derivative associated to the Dunkl connection $\omega = \varpi + \lambda$ is given by*

$$(3.7) \quad D_\omega(\varphi)(x) = D(\varphi)(x) + i \sum_{\alpha \in R^+} h_\alpha(x)(\varphi(x) - \varphi(x\sigma_\alpha))\alpha$$

where $D: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$ is the standard De Rham derivative of classical differential forms on P .

Proof. This follows directly from the general expression for covariant derivatives of connections as presented in the appendices. \square

We can now use the following ansatz to generate an interesting class of Dunkl connections:

$$(3.8) \quad h_\alpha(x) = \psi_\alpha[\langle \alpha, x \rangle],$$

where $\psi_\alpha: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ are smooth, *odd* functions indexed by the elements of $\alpha \in R$ in a G -invariant way (so effectively they are indexed by orbits of the action of G on R).

Note that for all $\alpha \in R$ the scalar products $\langle x, \alpha \rangle$ are never zero, because $x \in P$. As a very particular case of the above expression, we can consider for $r \in \mathbb{R} \setminus \{0\}$

$$(3.9) \quad \psi_\alpha(r) = \varkappa_\alpha \frac{1}{r},$$

where $\varkappa: R \rightarrow \mathbb{C}$ is a (necessarily G -invariant and, for the purpose of having a real connection, real-valued) multiplicity function defined on R . With such a choice we

will be able to reproduce the standard expression for the Dunkl operators as given in the previous section.

Definition 3.4. We call the connection associated to the choice (3.9) for ψ_α the *standard Dunkl connection*.

Indeed, the formula for the covariant derivative gives in this case

$$(3.10) \quad D_\omega(\varphi)(x) = D(\varphi)(x) + i \sum_{\alpha \in R^+} \kappa_\alpha \frac{\varphi(x) - \varphi(x\sigma_\alpha)}{\langle x, \alpha \rangle} \alpha,$$

which for functions corresponds to the classical definition up to a factor of i .

Remark 3.5. This factor of i corresponds to our geometric condition of reality of the connection. However, from the point of view of the algebraic considerations we develop here this is inessential. See the concluding section for more details.

It is worth mentioning that we can also use the more general ansatz

$$(3.11) \quad h_\alpha(x) = \psi_\alpha[\langle x, \alpha \rangle, \langle x, x \rangle],$$

where now ψ_α are smooth functions of two variables and the indexing by the α 's is done as before in a G -invariant way.

We can consider situations where there is no indexing at all in the sense that

$$\psi_\alpha(u, v) = \psi(u, v)$$

for a single smooth function ψ . It turns out that in this case the ansatz (3.11) captures all ‘generic’ displacements. By generic displacements, we understand those that are naturally associated, by using (3.13) below, to $O(E)$ -covariant maps $\mu: \text{RP}(E) \rightarrow \Lambda^1(E)$ defined on the real projective space $\text{RP}(E)$ associated to E . Also $\Lambda(E)$ is the algebra of classical complex differential forms on E . In other words, we assume that

$$(3.12) \quad \mu(gpg^{-1}) = \mu(p)_g$$

for every $g \in O(E)$ and $p \in \text{RP}(E)$. The elements of $\text{RP}(E)$ are interpreted as orthogonal projectors onto the one-dimensional subspaces of E . Generic displacements λ are characterized by

$$(3.13) \quad \lambda[\sigma_\alpha] = \mu(p_\alpha)$$

where $p_\alpha = (1 - \sigma_\alpha)/2$ is the projector onto the space $\mathbb{R}\alpha$. Without loss of generality, we can assume that all vectors α are normalized as in Definition 2.1, part 4.

Proposition 3.6. (i) If λ is a generic displacement (in particular, real in the sense of intertwining the corresponding $*$ -structures), then for any given $\alpha \in R$ we have that $\xi = -i\lambda[\sigma_\alpha]$ is a real one-form, invariant under σ_α and also invariant under the group of all orthogonal transformations that have α as a fixed point. (Let us call this group $O(\alpha, E)$.)

(ii) Conversely, if $\xi = \xi^*$ is a one-form on E invariant under σ_α and under all transformations from $O(\alpha, E)$, then there exists a unique displacement λ such that $\lambda[\sigma_\alpha] = i\xi$.

(iii) Every such one-form can be naturally decomposed (in the region outside of the line $\mathbb{R}\alpha$) as

$$(3.14) \quad \xi(x) = \psi\{\langle x, \alpha \rangle, \langle x, x \rangle\} \alpha + \tilde{\psi}\{\langle x, \alpha \rangle, \langle x, x \rangle\} \zeta$$

where $\psi, \tilde{\psi}: \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ are smooth functions satisfying

$$(3.15) \quad \psi(u, v) = -\psi(-u, v) \quad \tilde{\psi}(u, v) = \tilde{\psi}(-u, v)$$

and ζ is the canonical radial one-form on E . Moreover, the displacement λ will be closed in the standard sense of $D\lambda = 0$ if and only if

$$(3.16) \quad \frac{\partial \psi}{\partial v} = -\frac{\partial \tilde{\psi}}{\partial u}.$$

Proof. If λ is a generic displacement, then the covariance formula (3.12) implies that the reflection σ_α as well as the group $O(\alpha, E)$ act trivially on $\xi = -i\lambda[\sigma_\alpha]$. The elements ξ are real one-forms because λ , being a connection displacement, commutes with the corresponding $*$ -structures (and the $[\sigma_\alpha]$ are *imaginary* in Γ_{inv}). And conversely, if we fix $\alpha \in R$ and a real σ_α -invariant and $O(\alpha, E)$ -invariant one-form ξ on E , then every generic displacement λ satisfying $\lambda[\sigma_\alpha] = \mu(p_\alpha) = -i\xi$ should also satisfy

$$\lambda[\sigma_{\alpha g^{-1}}] = \mu(p_{\alpha g^{-1}}) = \mu(gp_\alpha g^{-1}) = \mu(p_\alpha)_g = -i\xi_g,$$

where $g \in O(E)$ is such that $\alpha g^{-1} \in R$. This shows that such a displacement is unique, if it exists. (The orthogonal group acts transitively on $RP(E)$ and in particular we can obtain all root vectors by acting on α via appropriate orthogonal transformations). What remains to prove is that the above formula consistently defines the displacement λ . If $h \in O(E)$ is another element such that $\alpha h^{-1} = \pm \alpha g^{-1}$ then either $g^{-1}h \in O(\alpha, E)$ or $g^{-1}h \in \sigma_\alpha O(\alpha, E)$. Therefore

$$\xi_h = \xi_{g(g^{-1}h)} = (\xi_{g^{-1}h})_g = \xi_g,$$

which shows the consistency of the definition formula for λ . The formula (3.14) is a generic expression for $O(\alpha, E)$ -invariant one-forms. Indeed, decomposing E as

$$E = \mathbb{R}\alpha \oplus (\mathbb{R}\alpha)^\perp$$

we see that ξ must be an appropriate combination of $\zeta - (x, \alpha)\alpha/2$ which is the canonical radial form in $\mathbb{R}\alpha^\perp$, and α which corresponds to the first summand $\mathbb{R}\alpha$ in the above decomposition. In other words, we have

$$\xi(x) = q(x)\alpha + \tilde{q}(x)\zeta,$$

where q and \tilde{q} are smooth functions on $E \setminus \mathbb{R}\alpha$. (We excluded the line $\mathbb{R}\alpha$ where the forms α and x are proportional in order to have a unique factorization). In order to preserve the $O(\alpha, E)$ -symmetry the functions q and \tilde{q} can only depend on the first coordinate as well as the radial part of the second coordinate in the above decomposition. To put it in equivalent terms, we obtain

$$q(x) = \psi(\langle x, \alpha \rangle, \langle x, x \rangle), \quad \tilde{q}(x) = \tilde{\psi}(\langle x, \alpha \rangle, \langle x, x \rangle).$$

Equations (3.15) are equivalent to the σ_α -invariance of ξ . Indeed the radial one-form ζ is σ_α -invariant, while α changes sign under the reflection by σ_α . Finally (3.16) is a matter of direct calculation of the differential, taking into account $D\langle x, \alpha \rangle = \alpha$ and $D\langle x, x \rangle = 2\zeta$. \square

Remark 3.7. Due to covariance we see that all generic displacements are of the form

$$(3.17) \quad \lambda[\sigma_\alpha](x) = i\psi(\langle \alpha, x \rangle, \langle x, x \rangle)\alpha + i\tilde{\psi}(\langle \alpha, x \rangle, \langle x, x \rangle)x,$$

where $\alpha \in R$ is now an arbitrary root vector.

So we can get (3.8) as an important special case (with a constant weight function, since all root vectors are equivalent in this picture) when ψ does not depend on the radial variable v and $\tilde{\psi} = 0$. Another special case is when $\psi = 0$ and $\tilde{\psi}$ does not depend on u . In the latter case we obtain a spherically symmetrical displacement λ taking the same value on all of the generators $[\sigma_\alpha]$.

Of course, an arbitrary displacement need not be generic. With the help of local trivializations of the bundle P , as indicated in Appendix B, we can easily classify all possible displacements, in terms of their local representations.

Here is a version of the above proposition for arbitrary displacements. The only symmetries we have are those of the Coxeter group G , and in general this group does not act transitively on R . For each $\alpha \in R$ we let G_α be the stabilizer of α in G , and we let O_α be the orbit of α under the action of G . Clearly $O_{-\alpha} = O_\alpha$ and we have a natural identification $G/G_\alpha \leftrightarrow O_\alpha$ between O_α and the right cosets of G_α in G .

Proposition 3.8. (i) *If λ is an arbitrary displacement, then $\xi_\alpha = -i\lambda[\sigma_\alpha]$ is real and invariant under σ_α as well as under all of the elements in G_α .*

(ii) *Conversely, let us choose for every orbit O of the action of G in R one element $\alpha \in O$. (So that we have $O = O_\alpha$.) Let us assume that for every such representative α a real one-form ξ_α on P is given which is invariant under σ_α and under all of the transformations in G_α . Then there exists a unique displacement λ such that $\lambda[\sigma_\alpha] = i\xi_\alpha$ for every representing root vector α .*

(iii) *Moreover, the displacement λ will be closed if and only if*

$$(3.18) \quad \xi_\alpha = D\psi_\alpha$$

for each $\alpha \in R$, where ψ_α are smooth real functions on P invariant under G_α and under σ_α . We can naturally make the system of functions $\{\psi_\alpha\}_{\alpha \in R}$ covariant in the sense that

$$(3.19) \quad \psi_{\alpha g}(x) = \psi_\alpha(xg^{-1})$$

for every $x \in P$ and $g \in G$.

Proof. To prove (i) and (ii) we can follow conceptually the same reasoning as for the corresponding properties of the previous proposition. If the forms $\lambda[\sigma_\alpha]$ are closed, then due to the contractability of the space P , they are globally exact. It is worth mentioning that due to the covariance of λ it is sufficient to check closedness for one representative of each orbit O . Finally, the possibility of choosing a G -covariant system $\{\psi_\alpha\}_{\alpha \in R}$ is a direct consequence of the G -covariance of the differential D and the fact that a short exact sequence of G -modules always splits. (This fact holds for arbitrary compact groups, whether classical or quantum.) \square

Remark 3.9. So we have a kind of *elementary* displacement where we pick out a single orbit O and one element $\alpha \in O = O_\alpha$ and then proceed by constructing a one-form ξ_α . For the representatives of the other orbits we can put $\xi_\alpha = 0$. In other words the associated displacements will vanish on all elements $[\sigma_\beta]$ if $\beta \notin O$.

Let us now calculate the curvature of the displaced connection $\omega = \varpi + \lambda$. Let us recall that, according to the general formalism and as explained in Appendix A,

this curvature is related to the original curvature via

$$(3.20) \quad r_\omega(a) = r_\varpi(a) + D_\varpi \lambda \pi(a) + \lambda \pi(a^{(1)}) \lambda \pi(a^{(2)}) + \ell_\varpi(\pi(a^{(1)}), \lambda \pi(a^{(2)})) \\ = D \lambda \pi(a) + \lambda \pi(a^{(1)}) \lambda \pi(a^{(2)}).$$

In our case, the initial connection ϖ is the canonical flat connection, so we have $D_\varpi = D$ and $r_\varpi = 0$. By construction, the canonical flat connection is *regular* so we also have $\ell_\varpi = 0$.

Our main goal now is to prove that under certain general conditions, the curvature of the displaced connection vanishes too. These general conditions will include as a special case the classical Dunkl operators. This means that it suffices to prove that the remaining terms in the above expression (3.20) vanish identically for appropriate displacements λ . We shall also derive an explicit formula for calculating the curvature of a general Dunkl connection (where the third term is the only one that possibly does not vanish).

Lemma 3.10. *For a Dunkl connection given by*

$$(3.21) \quad \lambda[\sigma_\alpha](x) = i\psi_\alpha[\langle x, \alpha \rangle] \alpha$$

we have $D\lambda = 0$. In other words, the displacement λ is closed as a standard differential form on P .

This is just a convenient rephrasing of the previously established result. It is worth remembering that in our quantum differential calculus, classical differential forms on P play the role of horizontal differential forms for the complete calculus on the bundle P .

So the curvature of such a Dunkl connection is given by the third term in the expression for the displaced curvature.

Definition 3.11. An element ρ of the Coxeter group G is called a *2-rotation* if $\rho = \sigma_\alpha \sigma_\beta$ for some $\alpha, \beta \in R$ and $\alpha \neq \pm\beta$. Such a rotation is called *proper* if its period is greater than two (in other words α and β are not orthogonal).

For a given proper 2-rotation ρ we can introduce in a canonical way a horizontal 2-form w_ρ which we can interpret as the volume element of the plane of the rotation of ρ . Indeed if we define

$$(3.22) \quad w_\rho = \alpha \wedge \beta \quad \text{where} \quad \rho = \sigma_\alpha \sigma_\beta,$$

then this formula determines w_ρ up to a sign. We can fix the ambiguity of the sign by requiring that α and β follow the orientation of the plane induced by ρ . It is worth observing that

$$(3.23) \quad w_{\rho^{-1}} = -w_\rho.$$

Proposition 3.12. (i) *Let G be a Coxeter group. Then the curvature r_ω of any connection $\omega = \varpi + \lambda$ vanishes on all elements of G except possibly some proper two-dimensional rotations ρ in which case we have*

$$(3.24) \quad r_\omega(\rho) = \sum^* \lambda[\sigma_\alpha] \lambda[\sigma_\beta].$$

Here the sum \sum^ is taken over all possible decompositions $\rho = \sigma_\alpha \sigma_\beta$ into the product of two reflections in G .*

(ii) Moreover, if ω is a Dunkl connection, then we have

$$(3.25) \quad r_\omega(\rho) = -w_\rho \sum^* h_\alpha h_\beta,$$

where the root vectors participating in the above sum are such that $\alpha \wedge \beta = w_\rho$.

Proof. Let us begin our proof by recalling that

$$\phi(g) = g^{(1)} \otimes g^{(2)} = \sum_{h \in G} h \otimes (h^{-1}g)$$

for each $g \in G$. On the other hand, by the construction of the differential calculus Γ over G , the map π vanishes on all elements except the reflections $s = \sigma_\alpha$ and the neutral element ϵ , where we have $[\epsilon] = -\sum_{s \in S} [s]$.

So the expression

$$\delta(g) = (\pi \otimes \pi)\phi(g) = \pi(g^{(1)}) \otimes \pi(g^{(2)})$$

vanishes on all $g \in G$ except possibly for the following three types of elements: The neutral element $\epsilon \in G$, in which case

$$\delta(\epsilon) = [\epsilon] \otimes [\epsilon] + \sum_{s \in S} [s] \otimes [s],$$

the reflections $s \in S$, where we have

$$\delta(s) = [\epsilon] \otimes [s] + [s] \otimes [\epsilon],$$

and finally

$$\delta(\rho) = \sum^* [\sigma_\alpha] \otimes [\sigma_\beta],$$

where ρ is 2-rotation and the sum is taken over all decompositions $\sigma_\alpha \sigma_\beta = \rho$.

The first two types of expressions are symmetric elements of $\Gamma_{inv} \otimes \Gamma_{inv}$ and hence, after being coupled with the classical one-form λ and multiplication in the exterior algebra of P , they vanish identically. Regarding the third and the most interesting type of potentially non-zero expressions, namely those associated with the 2-rotations, let us observe that if ρ is involutive (in other words given by the composition of two reflections associated to mutually orthogonal root vectors α and β), then $\delta(\rho)$ will also be symmetric because the term $[\sigma_\alpha] \otimes [\sigma_\beta]$ will appear in the sum if and only if its pair $[\sigma_\beta] \otimes [\sigma_\alpha]$ also appears, since the reflections σ_α and σ_β commute in this case.

So now the possibly non-zero terms in the formula for the curvature can only come from *proper* 2-rotations ρ . The curvature is therefore explicitly given by the third term of the decomposition (3.20), namely

$$r_\omega[\rho] = \sum^* \lambda[\sigma_\alpha] \lambda[\sigma_\beta] \quad \text{where} \quad \delta(\rho) = \sum^* [\sigma_\alpha] \otimes [\sigma_\beta].$$

In particular when ω is a Dunkl connection, we have

$$(3.26) \quad r_\omega(\rho) = -w_\rho \sum^* h_\alpha h_\beta,$$

and the elements α and β figuring in the above sum are such that $\alpha \wedge \beta = w_\rho$. These vectors α and β all belong to the 2-plane of the rotation ρ , and actually the problem is naturally being reduced to the subgroup of the Coxeter group generated by all the reflections associated to the above mentioned plane. It is worth mentioning that Coxeter groups in dimension 2 coincide with the class of dihedral groups. \square

Remark 3.13. The fact that curvature forms of connections can only take non-zero values on proper 2-rotations is an interesting purely quantum phenomenon. By definition the rotations project to zero by the quantum germs map $\pi: \mathcal{A} \rightarrow \Gamma_{inv}$. So the curvature is not indexed by the internal degrees of freedom of the calculus on the group (as is the case in classical geometry, where the curvature tensor has values in the Lie algebra of the structure group) but by objects ρ ‘external’ to the calculus.

Let us now consider a very special case of standard Dunkl connections, where $h_\alpha(x) = \varkappa_\alpha / \langle x, \alpha \rangle$. As we shall now see the curvature is identically zero in this case. This follows from 1.7 Proposition (1) in [4]. But for the convenience of the reader, we will include an alternative proof. The key in establishing this result is the following interesting property of vectors in the Euclidean plane.

Lemma 3.14. *Let us consider a two-dimensional Euclidean vector space Π with $\varrho: \Pi \rightarrow \Pi$, a rotation of order $2m$, where $m \geq 2$. (In other words the rotation ϱ satisfies $\varrho^m = -I$, the central reflection map in E .) Then we have the identity*

$$(3.27) \quad \frac{1}{\langle x, v \rangle \langle x, v\varrho \rangle} + \cdots + \frac{1}{\langle x, v\varrho^{m-2} \rangle \langle x, v\varrho^{m-1} \rangle} = \frac{1}{\langle x, v\varrho^{m-1} \rangle \langle x, v \rangle},$$

where $v \in \Pi$ is a unit vector and $x \in \Pi$ an arbitrary vector not orthogonal to any of the vectors $v, v\varrho, \dots, v\varrho^{m-1}$.

Proof. Our proof is based on a simple recursive sequence of vectors in Π . Let $v \in \Pi$ be a non-zero vector. Assume that it is associated with a sequence of vectors $\{v_k\}_{k \geq 1}$ in Π in such a way that

$$(3.28) \quad v = c_{k+1}v_k - c_kv_{k+1}$$

for each $k \geq 1$ and for some sequence of real numbers $\{c_k\}_{k \geq 1}$ satisfying $c_1 = 1$. Then for each $k \geq 2$ the following identity holds when all the denominators are non-zero:

$$(3.29) \quad \frac{1}{\langle x, v \rangle \langle x, v_1 \rangle} + \frac{1}{\langle x, v_1 \rangle \langle x, v_2 \rangle} + \cdots + \frac{1}{\langle x, v_{k-1} \rangle \langle x, v_k \rangle} = \frac{c_k}{\langle x, v \rangle \langle x, v_k \rangle}.$$

(Actually, this identity also holds for $k = 1$ provided that we define $v_0 := v$.) This identity can easily be proved inductively by using the above recursive relation (3.28). We leave that proof to the reader.

Returning to the context of this lemma we define $v_k = v\varrho^k$ for every integer $k \geq 1$. Then (3.28) holds with $c_k = \sin(k\nu)/\sin(\nu)$ and ν is an angle of a primitive m -th root of the central reflection map in Π (for example $\nu = \pi/m$). Observing that $c_{m-1} = 1$ completes the proof. \square

Remark 3.15. For $k = 3$ our identity can be derived from the following simple expression involving three arbitrary vectors $\alpha, \beta, \gamma \in \Pi$, namely

$$(3.30) \quad \frac{1}{\langle x, \alpha \rangle \langle x, \beta \rangle} + \frac{1}{\langle x, \beta \rangle \langle x, \gamma \rangle} + \frac{1}{\langle x, \gamma \rangle \langle x, \alpha \rangle} = \frac{\langle x, \alpha + \beta + \gamma \rangle}{\langle x, \alpha \rangle \langle x, \beta \rangle \langle x, \gamma \rangle},$$

provided that all the denominators are non-zero.

Theorem 3.16. *If ω is a standard Dunkl connection, then the curvature r_ω vanishes identically.*

Proof. Applying the formula (3.25) it is easy to see that the sum (3.26) reduces to a linear combination of one or more expressions each of which is a sum of the type

$$\left(\frac{1}{\langle x, v \rangle \langle x, v \varrho \rangle} + \cdots + \frac{1}{\langle x, v \varrho^{m-2} \rangle \langle x, v \varrho^{m-1} \rangle} \right) - \frac{1}{\langle x, v \varrho^{m-1} \rangle \langle x, v \rangle}.$$

It is important to note that the rotation ϱ is by an angle that is one half the angle of the rotation $\rho \in G$. According to (3.27) all these expressions are zero, and thus the curvature r_ω vanishes identically. \square

A very interesting modification of standard Dunkl connections arises if we use

$$h_\alpha(x)/\varkappa_\alpha = \coth[\langle x, \alpha \rangle / 2] = \frac{e^{\langle x, \alpha \rangle} + 1}{e^{\langle x, \alpha \rangle} - 1}.$$

The standard Dunkl connections can be viewed as first-order approximations of these connections. The curvature of such connections will not vanish on proper 2-rotations. However there exists a particular context where the curvature will have constant values on these rotations.

Let us first observe that in the above expression the lengths of the root vectors α matter in a very essential way (although as we mentioned, in the limit of small vectors x the formula reduces to the standard Dunkl form).

A particularly interesting context arises when we start from the vectors as in (3.28) and put $c_k = 1$ for all $k \geq 1$. In other words we have

$$(3.31) \quad v_k - v_{k+1} = v,$$

and so v_k is an arithmetic sequence of vectors, namely $v_{k+1} = v_1 - kv$ for all integers $k \geq 1$. We can then apply the following elementary operation: between each two neighbors of the sequence insert their sum.

Definition 3.17. Such a sequence will be called a *sequence of arithmetic type*.

Lemma 3.18. *The following identity holds*

$$(3.32) \quad \sum_{j=0}^k \coth[\langle x, v_j \rangle] \coth[\langle x, v_{j+1} \rangle] = \coth[\langle x, v \rangle] \coth[\langle x, v_{k+1} \rangle] + k$$

for every sequence $v_0 = v, v_1, \dots, v_k, v_{k+1}$ of vectors of arithmetic type.

Proof. The standard addition formula for the hyperbolic cotangent, namely

$$\coth(a + c) = \frac{1 + \coth(a)\coth(c)}{\coth(a) + \coth(c)},$$

gives

$$\coth[\langle x, \alpha \rangle] \coth[\langle x, \beta \rangle] + \coth[\langle x, \beta \rangle] \coth[\langle x, \gamma \rangle] = \coth[\langle x, \alpha \rangle] \coth[\langle x, \gamma \rangle] + 1,$$

where $\beta = \alpha + \gamma$ and we substitute $a = \langle x, \alpha \rangle$, $c = \langle x, \gamma \rangle$ and $b = \langle x, \beta \rangle = a + c$. Our main formula now easily follows by applying induction on k . \square

Our next natural and interesting question is whether sequences of arithmetic type appear in the context of calculating the curvature of Dunkl connections. It turns out that this is equivalent to assuming that the root system R is *classical* in the sense of being associated to a compact semisimple Lie algebra.

Proposition 3.19. (i) *Let us assume that the vectors $\beta + k\alpha$ for $k = 0, \dots, m$ (where $m \geq 1$ is an integer) and α generate a sequence of arithmetic type which forms a positive half of a two-dimensional root system. Then this system is classical and isomorphic to one of the following:*

- *The Hexagonal system for $m = 1$. In this case α and β have the same length, and the angle between them is $2\pi/3$.*
- *The Octagonal system. This corresponds to $\|\beta\| = \sqrt{2}\|\alpha\|$ and $m = 2$, or $\|\alpha\| = \sqrt{2}\|\beta\|$ and $m = 1$, with the angle between them being $3\pi/4$.*
- *The Dodecagonal snowflake system. In this case $\|\beta\| = \sqrt{3}\|\alpha\|$ and $m = 3$, or $\|\alpha\| = \sqrt{3}\|\beta\|$ and $m = 1$, and the angle between α and β is $5\pi/6$.*

In particular, the numbers 1, 2 and 3 are the only possible values for m .

(ii) *If the root system R is such that the above phenomenon occurs for every irreducible 2-dimensional subsystem of R , then R is classical.*

Proof. Everything follows by applying standard considerations for classifying 2-dimensional classical root systems [20]. Let us note in passing that in the case of the hexagonal root system a set of positive vectors is given by the initial sequence of the three vectors α , β and $\beta + \alpha$. In the case of the dodecagonal system, there always appear additional vectors, and for the octagonal system the additional vectors appear only when $\|\alpha\| = \sqrt{2}\|\beta\|$. \square

Remark 3.20. Let us observe that since the system R is classical we do not assume any longer the normalization condition, according to which all vectors are of the same norm equal to the square root of two. In fact, for standard Dunkl connections the norm of the vectors is irrelevant, since they appear linearly in the nominator and the denominator. In general situations however, it might be more convenient to use some particular normalization. In those cases, we can write a ‘neutral’ formula

$$(3.33) \quad w_\rho = 2 \frac{\alpha \wedge \beta}{\|\alpha\| \|\beta\|}$$

for the volume element of a proper two-dimensional rotation ρ . Clearly, this generalizes our previous definition.

Definition 3.21. We say that a Dunkl connection $\omega = \varpi + \lambda$ has *constant curvature* if the values of r_ω on proper 2-rotations ρ are of the form

$$(3.34) \quad r_\omega(\rho) = -c_\rho w_\rho,$$

where c_ρ are constants. In other words $c_\rho = \sum^* h_\alpha(x) h_\beta(x)$ does not depend on $x \in P$.

Theorem 3.22. *If the root system R is classical and a Dunkl connection $\omega = \varpi + \lambda$ is given by taking*

$$(3.35) \quad h_\alpha(x)/\varkappa_\alpha = \coth[\langle x, \alpha \rangle / 2],$$

then it has constant curvature.

Proof. This follows by a direct summation and by applying formulas (3.25) and (3.32). The number of terms in the defining sum for the curvature is half the number of elements in the corresponding 2-dimensional root subsystem generated by the rotation ρ . \square

As explained in Appendix A, general covariant derivatives satisfy a kind of twisted Leibniz rule with the presence of a third term involving the non-regularity measure operator ℓ_ω . Let us now calculate explicitly this operator for connections $\omega = \varpi + \lambda$ given by arbitrary displacements λ .

Lemma 3.23. *We have*

$$(3.36) \quad \ell_\omega([s], \varphi) = \lambda[s](\varphi - \varphi_s)$$

for every reflection $s \in G$ and every $\varphi \in \mathfrak{hor}(P)$.

Proof. A direct computation gives

$$\begin{aligned} \ell_\omega([s], \varphi) &= [s]\varphi - (-)^{\partial\varphi} \varphi^{(0)}([s] \circ \varphi^{(1)}) + \lambda[s]\varphi - (-)^{\partial\varphi} \varphi^{(0)}\lambda([s] \circ \varphi^{(1)}) \\ &= \lambda[s](\varphi - \varphi_s), \end{aligned}$$

where we have used the expansion $\varphi^{(0)} \otimes \varphi^{(1)} = \sum_{g \in G} \varphi_g \otimes g$, the definition of the product in $\Omega(P)$ and the fact that the right \mathcal{A} -module structure \circ acts on the elements $[s]$ as the scalar multiplication by the values of functions over G in these reflections, that is

$$[s] \circ f = f(s)[s]$$

for every $f \in \mathcal{A}$. Also when we use an expression such as $(-)^{\partial f}$ it means that f is a homogeneous element of a graded object and that ∂f is its degree. \square

Remark 3.24. In the above lemma, it was not necessary to assume that the calculus is based on reflections. In fact, the lemma holds for an arbitrary bicovariant and $*$ -covariant calculus over G . In particular we see that the connection based on a non-zero displacement λ is never a regular connection. Besides regular connections, there is another important class of connections, called *multiplicative connections*. Their defining property is (see [8] and Appendix A) the quadratic identity

$$\omega\pi(r^{(1)})\omega\pi(r^{(2)}) = 0$$

for every $r \in \mathcal{R}$, where $\mathcal{R} \subseteq \ker(\epsilon)$ is the right \mathcal{A} -ideal that determines the calculus Γ over G . Note that the elements $\pi(r^{(1)}) \otimes \pi(r^{(2)})$ provide precisely the quadratic relations for the universal differential envelope Γ^\wedge for Γ (as discussed in detail in [7], Appendix B), and in particular the algebra of left-invariant differential forms Γ_{inv}^\wedge can be obtained by dividing the tensor algebra Γ_{inv}^\otimes by these quadratic relations. Consequently, the above quadratic identity means precisely that ω is extendible to a multiplicative unital homomorphism $\omega^\wedge: \Gamma_{inv}^\wedge \rightarrow \Omega(P)$. This justifies the terminology ‘multiplicative connection’. It is also worth observing that for a general connection the expression $\omega\pi(r^{(1)})\omega\pi(r^{(2)})$ will always be horizontal for $r \in \mathcal{R}$. This is yet another purely quantum phenomenon, where we have expressed something horizontal as a homogeneous quadratic polynomial involving vertical objects only.

Now we can relate the multiplicativity of connections with vanishing curvature.

Proposition 3.25. *For a general connection $\omega = \varpi + \lambda$ the following properties are equivalent:*

- (i) *The connection is multiplicative or in other words there exists a (necessarily unique and $*$ -preserving) unital multiplicative extension $\omega^\wedge: \Gamma_{inv}^\wedge \rightarrow \Omega(P)$ of ω .*
- (ii) *The curvature of ω vanishes on \mathcal{R} or in other words $r_\omega(\rho) = 0$ for every two-dimensional rotation ρ .*

Proof. First, it is easy to see that the only elements $r \in \mathcal{R}$ for which the expression $\pi(r^{(1)}) \otimes \pi(r^{(2)})$ can possibly be non-zero are the two-dimensional rotations $r = \rho$. Second, a couple of elementary transformations leads us to conclude that

$$\begin{aligned} \lambda\pi(\rho^{(1)})\varpi\pi(\rho^{(2)}) + \varpi\pi(\rho^{(1)})\lambda\pi(\rho^{(2)}) &= \lambda\pi(\rho^{(1)}) \otimes \pi(\rho^{(2)}) \\ &\quad - \lambda\pi(\rho^{(3)}) \otimes [\pi(\rho^{(1)}) \circ (\kappa(\rho^{(2)})\rho^{(4)})] = \lambda\pi(\rho^{(2)}) \otimes \pi(\kappa(\rho^{(1)})\rho^{(3)}) = 0 \end{aligned}$$

because of the ad-invariance of \mathcal{R} or in other words $\text{ad}(\mathcal{R}) \subseteq \mathcal{R} \otimes \mathcal{A}$. Together with the obvious identity $\varpi\pi(\rho^{(1)})\varpi\pi(\rho^{(2)}) = 0$ this implies that

$$(3.37) \quad \omega\pi(\rho^{(1)})\omega\pi(\rho^{(2)}) = \lambda\pi(\rho^{(1)})\lambda\pi(\rho^{(2)}),$$

which is effectively the third term in the expression for the curvature of the displaced connection. \square

Remark 3.26. So when either of the equivalent conditions of the above lemma holds, the curvature behaves more ‘classically’, namely its internal indices are given by the basis of Γ_{inv} , and so it is naturally projectable down to Γ_{inv} . In this case we simply have

$$r_\omega(s) = D\lambda[s]$$

for every $s \in S$. In particular, the curvature vanishes if and only if the displacement λ is closed.

Remark 3.27. Let us also observe that the above lemma holds for an arbitrary quantum principal bundle with the differential calculus constructed as a crossed product of $\mathfrak{hot}(P)$ and Γ_{inv}^\wedge in the way described in Appendix A.

Using the above formula for the non-regularity obstacle ℓ_ω , we can find an explicit expression for the third term in the quantum Leibniz rule for covariant derivatives.

Theorem 3.28. *Let λ be an arbitrary displacement map. Then the associated connection $\omega = \varpi + \lambda$ satisfies*

$$(3.38) \quad D_\omega(\varphi\psi) = D_\omega(\varphi)\psi + (-)^{\partial\varphi}\varphi D_\omega(\psi) - \sum_{s \in S} \lambda[s](\varphi - \varphi_s)(\psi - \psi_s)$$

for all $\varphi, \psi \in \mathfrak{hot}(P)$.

Proof. A direct computation, using (B.11), gives

$$\begin{aligned} D_\omega(\varphi\psi) &= D_\omega(\varphi)\psi + (-)^{\partial\varphi}\varphi D_\omega(\psi) + (-)^{\partial\varphi}\varphi^{(0)}\ell_\omega(\pi(\varphi^{(1)}), \psi) \\ &= D_\omega(\varphi)\psi + (-)^{\partial\varphi}\varphi D_\omega(\psi) + (-)^{\partial\varphi}\varphi\ell_\omega([\epsilon], \psi) + (-)^{\partial\varphi}\sum_{s \in S} \varphi_s\ell_\omega([s], \psi) \\ &= D_\omega(\varphi)\psi + (-)^{\partial\varphi}\varphi D_\omega(\psi) + (-)^{\partial\varphi}\sum_{s \in S} (\varphi_s - \varphi)\ell_\omega([s], \psi) \\ &= D_\omega(\varphi)\psi + (-)^{\partial\varphi}\varphi D_\omega(\psi) + (-)^{\partial\varphi}\sum_{s \in S} (\varphi_s - \varphi)\lambda[s](\psi - \psi_s). \end{aligned}$$

Besides elementary transformations, we have applied Lemma 3.23 and the general Leibniz rule for covariant derivatives. \square

Remark 3.29. This result generalizes to horizontal forms Theorem 2.4, which is only for functions.

We now apply these results regarding arbitrary displacements to Dunkl connections, where the displacements are of the form (3.8). In this special case the classical Leibniz rule is fulfilled for a large class of horizontal forms in conjunction with arbitrary forms. As we can see explicitly from (3.38) if one of the forms φ or ψ is G -invariant, in other words if it belongs to the forms on the base space M , then the classical Leibniz rule holds.

Proposition 3.30. *Let $\omega = \varpi + \lambda$ be a Dunkl connection.*

(i) *If θ is a coordinate one-form on P then*

$$(3.39) \quad D_\omega(\theta) = 0$$

and more generally we have

$$(3.40) \quad D_\omega(\varphi\theta) = D_\omega(\varphi)\theta$$

for each $\varphi \in \mathfrak{hor}(P)$.

(ii) *Let \mathcal{W} be the subalgebra of $\mathfrak{hor}(P)$ generated by the elements of $\Omega(M)$ and the coordinate one-forms θ . Then \mathcal{W} is $*$ -invariant, D -invariant and G -invariant in the sense that $F_\wedge(\mathcal{W}) \subseteq \mathcal{W} \otimes \mathcal{A}$. The action of the operator D_ω coincides with the action of D on \mathcal{W} . We have*

$$(3.41) \quad \begin{aligned} D_\omega(\varphi\psi) &= D_\omega(\varphi)\psi + (-)^{\partial\varphi}\varphi D(\psi) \\ D_\omega(\psi\varphi) &= D(\psi)\varphi + (-)^{\partial\psi}\psi D_\omega(\varphi) \end{aligned}$$

for each $\varphi \in \mathfrak{hor}(P)$ and $\psi \in \mathcal{W}$.

Proof. By definition, being differentials of coordinates, the coordinate one-forms θ are closed; in other words $D(\theta) = 0$. Now, the displacement $\lambda[s]$ for a Dunkl connection is proportional to the coordinate one-form α , where $s = \sigma_\alpha$ and the difference $\theta - \theta_s$ is also proportional to α . This means that the displacement part of the covariant derivative vanishes too, and hence (3.39) holds. Similarly, applying (3.38) we conclude that (3.40) holds. As a direct consequence of this property, we conclude that D and D_ω coincide on \mathcal{W} .

Finally, the properties in (3.41) follow from observing that they hold for ψ either in $\Omega(M)$ or equal to a coordinate form (and for arbitrary horizontal form φ). Then we use the fact that all such horizontal forms ψ satisfying the classical Leibniz rule for arbitrary $\varphi \in \mathfrak{hor}(P)$ form a subalgebra of $\mathfrak{hor}(P)$. \square

Remark 3.31. It is worth mentioning that, as a direct corollary of the above proposition, we have

$$(3.42) \quad D_\omega(\varphi\Theta) = D_\omega(\varphi)\Theta$$

for every Θ belonging to the 2^n -dimensional subalgebra generated by the coordinate one-forms.

We shall now analyze the coordinate representation of the covariant derivative maps and their relation with the quantum curvature tensor. In particular, we shall see that covariant directional derivatives commute among themselves when the curvature of a Dunkl connection vanishes.

Let us consider the canonical, Euclidean coordinates x_1, \dots, x_n as real-valued functions on P . For an arbitrary connection ω we have a natural decomposition

$$(3.43) \quad D_\omega(b) = \sum_{k=1}^n \partial_\omega^k(b) \theta_k \quad \theta_k = D(x_k),$$

where $b \in \mathcal{B}$. The linear maps $\partial_\omega^k: \mathcal{B} \rightarrow \mathcal{B}$, known as *covariant partial derivatives*, completely determine the operator D_ω . This is because a horizontal m -form φ can be decomposed naturally as

$$\varphi = \sum_{i_1 < \dots < i_m} b_{i_1 \dots i_m} \theta_{i_1} \dots \theta_{i_m}$$

with $b_{i_1 \dots i_m} \in \mathcal{B}$ and because D_ω is right linear over the subalgebra generated by the coordinate forms. Explicitly, for Dunkl connections these covariant partial derivatives are given for $k = 1, \dots, n$ by

$$(3.44) \quad \partial_\omega^k(b)(x) = \partial^k(b)(x) + i \sum_{\alpha \in R^+} \psi_\alpha[\langle x, \alpha \rangle] (b(x) - b(x\sigma_\alpha)) \alpha_k,$$

where $\alpha = (\alpha_1, \dots, \alpha_k, \dots, \alpha_n)$ and $\partial^k = \partial/\partial x_k$ is the usual partial derivative.

The coordinate representation of the curvature tensor is given by

$$(3.45) \quad r_\omega(\rho) = \frac{1}{2} \sum_{k,l=1}^n r_\omega^{kl}(\rho) \theta_k \theta_l,$$

where $r_\omega^{kl}(\rho) = -r_\omega^{lk}(\rho)$ are smooth functions on P and $\rho \in G$ an arbitrary proper two-dimensional rotation.

Proposition 3.32. *Let $\omega = \varpi + \lambda$ be a Dunkl connection.*

(i) *We have*

$$(3.46) \quad [\partial_\omega^k, \partial_\omega^l](b) + \sum_{\rho} b_\rho r_\omega^{kl}(\rho) = 0$$

for each $b \in \mathcal{B}$. Here the sum is taken over all proper 2-rotations ρ of the Coxeter group G .

(ii) *In particular, the curvature of the connection ω vanishes if and only if*

$$(3.47) \quad \partial_\omega^k \partial_\omega^l = \partial_\omega^l \partial_\omega^k$$

for every $k, l \in \{1, \dots, n\}$.

Proof. Let us recall a general expression connecting the curvature and the square of the covariant derivative:

$$D_\omega^2(\varphi) = -\varphi^{(0)} r_\omega(\varphi^{(1)}) = -\sum_{\rho} \varphi_\rho r_\omega(\rho),$$

where the sum on the right hand side runs over all of the proper two-dimensional rotations ρ , these being the only elements from G on which the curvature tensor

might possibly be non-zero. Using (3.43) and (3.40) as well as applying the above result about the curvature tensor for $b \in \mathcal{B}$, we obtain

$$\begin{aligned} D_\omega^2(b) &= - \sum_\rho b_\rho r_\omega(\rho) = -\frac{1}{2} \sum_\rho \sum_{k,l=1}^n b_\rho r_\omega^{kl}(\rho) \theta_k \theta_l = \sum_{k=1}^n D_\omega(\partial_\omega^k(b) \theta_k) \\ &= \sum_{k=1}^n D_\omega(\partial_\omega^k(b)) \theta_k = \frac{1}{2} \sum_{k,l=1}^n [\partial_\omega^k, \partial_\omega^l](b) \theta_k \theta_l. \end{aligned}$$

Hence (3.46) holds. In the special case when the curvature vanishes, we conclude that the partial covariant derivatives commute. Conversely, if the partial covariant derivatives commute, then the square of the covariant derivative vanishes on \mathcal{B} and thus it vanishes completely. This means that the curvature is zero. \square

Remark 3.33. It is worth recalling that in establishing the above properties, we have used in an essential way the fact that $\mathfrak{hor}(P)$ is the classical graded-commutative algebra of differential forms, and that the quantum connection ω is given by a very special kind of displacement map λ for which $\lambda[\sigma_\alpha]$ is always proportional to α . On the other hand, in the most general context of arbitrary quantum principal G -bundles the curvature r_ω may only have ‘purely quantum’ non-zero components on the neutral element where

$$r_\omega(\epsilon) = \lambda[\epsilon]^2 + \sum_{s \in S} \lambda[s]^2$$

and on two-dimensional rotations, including the non-proper ones of order two, besides the ‘standard’ components of r_ω on the elements $s \in S$ projecting down to the canonical basis in the space Γ_{inv} . On the 2-rotation elements we would have our already familiar expression

$$r_\omega(\rho) = \sum^* \lambda[\sigma_\alpha] \lambda[\sigma_\beta].$$

All of this is a consequence of the nature of the choice of the differential calculus on G . So we can consider it to be an interesting, purely quantum phenomenon that in the context of our main interest (where the bundle space is classical but the calculus is quantum and ω is a Dunkl connection) that the curvature can only assume non-zero values on proper rotations.

4. CONCLUDING OBSERVATIONS

The upshot of this article has been to view Dunkl operators in a much wider context than in previous research. That general context, quantum principal bundles with a given connection, is quite broad and we would like to draw attention to the particular niche that Dunkl operators occupy in it. The basic properties when viewed this way are the following:

- The spaces of the bundle are classical.
- The structural group of the bundle is classical, namely a finite Coxeter group, even though the theory allows in general any quantum group, finite or not.
- The differential calculi in the fiber space and in the total space are not classical, though the differential calculus in the base space is classical.

- The connection is a perturbation by a natural displacement term (that is determined by the choice of a particular subset of the Coxeter group) of the classical Levi-Civita connection of curvature zero in an open subset of a Euclidean space. This perturbed connection is not classical and has curvature zero.
- The covariant derivatives form a commutative family of operators.

We have used ‘classical’ here to refer to the context of classical differential geometry. The theory of quantum principal bundles allows the possibility of changes in all of these properties. As one example let us note that any finite group gives rise to a quantum group in the sense of Woronowicz. So we foresee many avenues for future research, both in geometry and analysis, based on this point of view.

In this paper we have assumed that the quantum connections $\omega: \Gamma_{inv} \rightarrow \Omega(P)$ are real in the sense of intertwining the $*$ -structures on Γ_{inv} and $\Omega(P)$. This condition corresponds to the standard reality property of the connection form in classical differential geometry. Because of it, we have the appearance of the imaginary unit $i = \sqrt{-1}$ in several expressions in contrast with the usual formulas in Dunkl theory. From the algebraic viewpoint this reality condition is not essential and the rest of our formalism works without it as well. Geometrically, this corresponds to playing with the elements of the complexified affine space of connections. Every such complex connection can be uniquely decomposed as $\omega_C = \omega + i\lambda$ where ω and λ are a standard (real) connection and a displacement map, respectively.

Our main example of a principal bundle P is not very interesting topologically, since it is a trivial bundle over the base M , which itself is the Weyl chamber associated to the root system R . Much more interesting topological situations [5] arise when we consider a complex Euclidean space (the complexification of the initial Euclidean space E).

Several generalizations of the original Dunkl theory have appeared over the years in the literature. We expect that these also will fit into the general, basic framework presented in this paper. Among them we mention the Dunkl-Cherednik operators [19], the Jacobi-Dunkl operators [1] and the complex Dunkl operators [5]. The last reference concerns a situation where the manifolds have complex structure, this being something well within the scope of our methods. In such a context we could deal with complex scalar products explicitly, and so with complex connections and displacements. Moreover, our main results and formulas would still remain valid in this context too.

Another interesting topic for further research would be to investigate arbitrary classical and quantum principal bundles whose structure group is a Coxeter group. This group can be equipped with the same differential calculus based on reflections as we have done in this paper or with any other different calculus as outlined in the general theory in the appendices.

APPENDIX A. QUANTUM PRINCIPAL BUNDLES

In this appendix we shall review general properties of quantum principal bundles with the emphasis on the formalism of connections and covariant derivatives. More detailed explanations and proofs can be found in [8, 9].

Let us first recall the basic definition and interpretation for a quantum principal bundle $P = (\mathcal{B}, \iota, F)$. The $*$ -algebra \mathcal{B} defines P as a quantum space, while the

*-algebra \mathcal{V} corresponds to the base quantum space M . There is a natural inclusion map $\iota: \mathcal{V} \hookrightarrow \mathcal{B}$ which can be interpreted as a ‘fibration’ of P over M . Furthermore there is a symmetry of P by a quantum group G via a linear map $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$. Strictly speaking, the dual concept of an action is a *co-action*, and so this is how one should refer to the map F . However, we will often speak of F and similar maps as actions, since classically they correspond by duality to honest-to-goodness actions. Also the base space algebra \mathcal{V} consists of all F -invariant elements of \mathcal{B} , namely,

$$\mathcal{V} = \{b \in \mathcal{B} \mid F(b) = b \otimes 1\}.$$

The differential calculus on P is specified by a graded differential *-algebra $\Omega(P)$ over \mathcal{B} . Let us denote by $d_P: \Omega(P) \rightarrow \Omega(P)$ the corresponding differential. We assume that there is a differential morphism $\widehat{F}: \Omega(P) \rightarrow \Omega(P) \widehat{\otimes} \Gamma^\wedge$, itself an action, extending the action F . Here Γ^\wedge is the enveloping differential algebra of a given bicovariant *-covariant first-order calculus Γ over G , as explained and constructed in [7]–Appendix B. We have explicitly these defining properties of an action:

$$(\text{id} \otimes \epsilon)\widehat{F} = \text{id} \quad (\widehat{F} \otimes \text{id})\widehat{F} = (\text{id} \otimes \widehat{\phi})\widehat{F},$$

where $\widehat{\phi}: \Gamma^\wedge \rightarrow \Gamma^\wedge \widehat{\otimes} \Gamma^\wedge$ is the graded differential extension of the coproduct ϕ of \mathcal{A} and the counit ϵ is extended trivially.

Alternatively, we can use the higher-order calculus of [23] based on the braided exterior algebra associated to Γ . These two higher order calculi are the maximal (the former) and the minimal (the latter) objects in the category of all higher-order calculi extending Γ for which the coproduct map extends to the graded-differential level (necessarily uniquely). It is also worth mentioning that such a property implies the bicovariance of Γ .

The intuitive geometrical interpretation of \widehat{F} and $\widehat{\phi}$ is that they are pullbacks at the level of differential forms of the right action $P \times G \rightsquigarrow P$ and the multiplication $G \times G \rightsquigarrow G$.

If a differential calculus $\Omega(P)$ satisfying the mentioned properties is given, then the algebra of horizontal forms $\mathfrak{hor}(P)$ can be defined as

$$\mathfrak{hor}(P) = \left\{ w \in \Omega(P) \mid \widehat{F}(w) \in \Omega(P) \otimes \mathcal{A} \right\},$$

and the restriction of \widehat{F} to $\mathfrak{hor}(P)$ induces a map $F_\wedge: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P) \otimes \mathcal{A}$. The interpretation is that the horizontal forms exhibit trivial differential properties along the vertical fibers of the bundle. In the spirit of this interpretation the calculus on the base quantum space M is given by F_\wedge -invariant elements of $\mathfrak{hor}(P)$ or, to put it in equivalent terms, by the \widehat{F} -invariant subalgebra $\Omega(M)$ of $\Omega(P)$. It is a differential *-subalgebra of the full calculus. We shall write $d_M: \Omega(M) \rightarrow \Omega(M)$ for the corresponding restricted differential.

It is worth observing that we have $\Omega^0(P) = \mathcal{B} = \mathfrak{hor}^0(P)$ and $\Omega^0(M) = \mathcal{V}$. As a differential algebra we assume that \mathcal{B} generates $\Omega(P)$. In other words, the n -th grade forms are expressible as

$$\Omega^n(P) = \left\{ \sum b d_P(b_1) \dots d_P(b_n) \right\} = \left\{ \sum d_P(b_1) \dots d_P(b_n) b \right\}.$$

Somewhat surprisingly $\Omega(M)$ will not in general be generated by \mathcal{V} , although in some very important special contexts this property will be fulfilled.

Let us now consider a connection $\omega: \Gamma_{inv} \rightarrow \Omega^1(P)$ on P . By definition ω is a linear map such that for all $\vartheta \in \Gamma_{inv}$ we have

$$\omega(\vartheta^*) = \omega(\vartheta)^*, \quad \widehat{F}\omega(\vartheta) = (\omega \otimes \text{id})\text{ad}(\vartheta) + 1 \otimes \vartheta.$$

The above conditions correspond to the classical idea [14] of a $\text{lie}(G)$ -valued pseudotensorial real one-form which maps fundamental vertical vector fields back into their generators. The adjoint action $\text{ad}: \Gamma_{inv} \rightarrow \Gamma_{inv} \otimes \mathcal{A}$ of G on Γ_{inv} is given by

$$\text{ad}\pi(a) = \pi(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)}$$

and $\pi: \mathcal{A} \rightarrow \Gamma_{inv}$ is the corresponding ‘quantum germs’ projection map $\pi(a) = \kappa(a^{(1)})d(a^{(2)})$. We have

$$\begin{aligned} \pi(ab) &= \epsilon(a)\pi(b) + \pi(a) \circ b \\ \pi(a)^* &= -\pi[\kappa(a)^*] \\ d\pi(a) &= -\pi(a^{(1)})\pi(a^{(2)}) \\ d(a) &= a^{(1)}\pi(a^{(2)}) \quad d\kappa(a) = -\pi(a^{(1)})\kappa(a^{(2)}) \end{aligned}$$

where \circ is the canonical right \mathcal{A} -module structure on Γ_{inv} . It is explicitly given by $\vartheta \circ a = \kappa(a^{(1)})\theta a^{(2)}$. We also have $\pi(a) \circ b = \pi(ab) - \epsilon(a)\pi(b)$ for all $a, b \in \mathcal{A}$.

The covariant derivative of ω is the map $D_\omega: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$ defined by

$$(A.1) \quad D_\omega(\varphi) = d_P(\varphi) - (-)^{\partial\varphi} \varphi^{(0)} \omega \pi(\varphi^{(1)}).$$

It is a kind of covariant perturbation of d_P by ‘vertical’ terms, so that the resulting expression becomes horizontal. In the same spirit we also define a ‘mirror’ derivative by moving the germs part to the left of φ and keeping everything horizontal, namely

$$(A.2) \quad D'_\omega(\varphi) = d_P(\varphi) + \{\omega \pi \kappa^{-1}(\varphi^{(1)})\} \varphi^{(0)}.$$

These two derivatives are related by the expression

$$(A.3) \quad D'_\omega(\varphi) = D_\omega(\varphi) + \ell_\omega(\pi \kappa^{-1}(\varphi^{(1)}), \varphi^{(0)}),$$

where $\ell_\omega: \Gamma_{inv} \times \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$ is the *regularity deviation* measure [10]. This is a kind of twisted commutator between connections and horizontal forms given by

$$(A.4) \quad \ell_\omega(\vartheta, \varphi) = \omega(\vartheta)\varphi - (-)^{\partial\varphi} \varphi^{(0)} \omega(\vartheta \circ \varphi^{(1)}).$$

Remark A.1. As explained in [8] and in much more detail in [10] and [9], there are very special connections, called *regular connections*, characterized by the equation $\ell_\omega = 0$, in other words, a kind of twisted commutation relation between a connection form and its horizontal forms. If a bundle admits a regular connection, then the theory of characteristic classes assumes a particularly simple form that is a braided version of classical Weil theory. In general, a connection will be not regular, and the operator ℓ_ω in a sense measures the deviation from regularity. It is interesting to observe that the values of ℓ_ω are always horizontal, in spite of the fact that it contains the vertical object ω . This is a purely quantum phenomenon. In classical geometry horizontal forms can not include in a non-trivial algebraic way the components of vertical objects such as connection forms.

Then the following identities hold:

$$(A.5) \quad \ell_\omega(\vartheta, \varphi\psi) = \ell_\omega(\vartheta, \varphi)\psi + (-)^{\partial\varphi}\varphi^{(0)}\ell_\omega(\vartheta \circ \varphi^{(1)}, \psi),$$

$$(A.6) \quad -\ell_\omega(\vartheta, \varphi)^* = \ell_\omega(\vartheta^* \circ \kappa(\varphi^{(1)})^*, \varphi^{(0)*}),$$

$$(A.7) \quad F^\wedge \ell_\omega(\vartheta, \varphi) = \ell_\omega(\vartheta^{(0)}, \varphi^{(0)}) \otimes \vartheta^{(1)}\varphi^{(1)}.$$

By construction the operators D_ω and D'_ω both extend the differential

$$d_M: \Omega(M) \rightarrow \Omega(M).$$

We have that

$$(A.8) \quad D_\omega(\varphi\psi) = D_\omega(\varphi)\psi + (-)^{\partial\varphi}\varphi D_\omega(\psi) + (-)^{\partial\varphi}\varphi^{(0)}\ell_\omega\{\pi(\varphi^{(1)}), \psi\},$$

$$(A.9) \quad D'_\omega(\varphi\psi) = D'_\omega(\varphi)\psi + (-)^{\partial\varphi}\varphi D'_\omega(\psi) + \ell_\omega\{\pi\kappa^{-1}(\psi^{(1)}) \circ \kappa^{-1}(\varphi^{(1)}), \varphi^{(0)}\}\psi^{(0)}.$$

These generalize the standard Leibniz rules. The restricted Leibniz rules

$$(A.10) \quad D_\omega(w\varphi) = d_M(w)\varphi + (-)^{\partial w}wD_\omega(\varphi)$$

$$(A.11) \quad D'_\omega(\varphi w) = D'_\omega(\varphi)w + (-)^{\partial\varphi}\varphi d_M(w)$$

hold for $w \in \Omega(M)$ and $\varphi \in \mathfrak{hor}(P)$. These two derivatives are mutually conjugate in the sense that

$$(A.12) \quad *D_\omega* = D'_\omega$$

The operator ℓ_ω is completely determined by the covariant derivative map D_ω since we have

$$(A.13) \quad \ell_\omega(\pi(a), \varphi) = [a]_1 D_\omega\{[a]_2 \varphi\} - [a]_1 D_\omega\{[a]_2\}\varphi,$$

where $\varphi \in \mathfrak{hor}(P)$ and $a \in \ker(\epsilon)$. Also we used the symbol $[a]_1 \otimes [a]_2$ to denote the value of the translation map $\tau: \mathcal{A} \rightarrow \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B}$ on $a \in \mathcal{A}$. (See [9] for more details.) It is also worth mentioning that the regularity property of ω (defined as the vanishing condition $\ell_\omega = 0$) is equivalent to the standard Leibniz rules for D_ω or D'_ω . If this is the case, then the covariant derivative is also hermitian, that is, $*D_\omega = D'_\omega*$.

Another fundamental object naturally associated to ω is its *curvature*. It can be expressed in terms of the square of the covariant derivative operator. The restricted Leibniz rules over $\Omega(M)$ imply that the square of the covariant derivative will always be a left and right $\Omega(M)$ -linear map. More precisely, we have

$$(A.14) \quad \begin{aligned} D_\omega^2(\varphi) &= -\varphi^{(0)}r_\omega(\varphi^{(1)}) \\ r_\omega(a) &= -[a]_1 D_\omega^2[a]_2 \end{aligned}$$

with the ‘curvature tensor’ $r_\omega: \mathcal{A} \rightarrow \mathfrak{hor}(P)$ given by a quantum version of the classical structure equation

$$(A.15) \quad r_\omega(a) = d_P\omega\pi(a) + \omega\pi(a^{(1)})\omega\pi(a^{(2)}).$$

It is instructive to compute directly the square of the covariant derivative. We have

$$\begin{aligned} D_\omega^2(\varphi) &= d_P D_\omega(\varphi) + (-)^{\partial\varphi} D_\omega(\varphi^{(0)})\omega\pi(\varphi^{(1)}) = d_P(d_P(\varphi) - (-)^{\partial\varphi}\varphi^{(0)}\omega\pi(\varphi^{(1)})) \\ &\quad + (-)^{\partial\varphi} d_P(\varphi^{(0)})\omega\pi(\varphi^{(1)}) - \varphi^{(0)}\omega\pi(\varphi^{(1)})\omega\pi(\varphi^{(2)}) = -\varphi^{(0)}d_P\omega\pi(\varphi^{(1)}) \\ &\quad - \varphi^{(0)}\omega\pi(\varphi^{(1)})\omega\pi(\varphi^{(2)}) = -\varphi^{(0)}r_\omega(\varphi^{(1)}) \end{aligned}$$

with r_ω given by the above structure equation. Furthermore, the following identities hold:

$$(A.16) \quad D_\omega r_\omega(a) + \ell_\omega\{\pi(a^{(1)}), r_\omega(a^{(2)})\} = 0,$$

$$(A.17) \quad r_\omega(a)^* = -r_\omega[\kappa(a)^*],$$

$$(A.18) \quad F^\wedge r_\omega(a) = r_\omega(a^{(2)}) \otimes \kappa(a^{(1)})a^{(3)}.$$

These generalize the corresponding formulas from classical differential geometry. In particular, the first of these identities is a quantum version of the classical Bianchi identity. The second identity is the reality property for the curvature, and the third establishes its transformation rule, which classically corresponds to the tensoriality property of the curvature tensor.

It is worth observing that in general the map r_ω will be not projectable down to quantum germs in contrast with the classical situation. So this is a purely quantum phenomenon which can be interpreted as a kind of ‘inadequacy’ of the calculus Γ for the bundle P .

We also have the following rule for covariantly differentiating the regularity obstacle:

$$(A.19) \quad D_\omega \ell_\omega(\vartheta, \varphi) = r_\omega(a)\varphi - \varphi^{(0)}r_\omega(a\varphi^{(1)}) - \ell_\omega(\vartheta, D_\omega(\varphi)) \\ - \ell_\omega[\pi(a^{(1)}), \ell_\omega(\pi(a^{(2)}), \varphi)].$$

In the above formula $\vartheta = \pi(a)$ and $a \in \ker(\epsilon)$.

Let us now consider a linear map $\lambda: \Gamma_{inv} \rightarrow \mathfrak{hor}^1(P)$ satisfying

$$\lambda(\vartheta^*) = \lambda(\vartheta)^* \quad F_\wedge \lambda(\vartheta) = (\lambda \otimes \text{id})\text{ad}(\vartheta)$$

for each $\vartheta \in \Gamma_{inv}$. Such maps can be interpreted as connection displacements. They form the vector space (in general, infinite dimensional) associated to the affine space of all connections.

So it is interesting to ask what is the relation between the covariant derivatives and curvature tensors of two connections ω and $\omega + \lambda$. We used these formulas in the main text to establish the principal properties of Dunkl operators. We have

$$(A.20) \quad r_{\omega+\lambda}(a) = r_\omega(a) + D_\omega \lambda\pi(a) + \lambda\pi(a^{(1)})\lambda\pi(a^{(2)}) + \ell_\omega(\pi(a^{(1)}), \lambda\pi(a^{(2)})).$$

Indeed, applying the structure equation (A.15) for the displaced connection $\omega + \lambda$, and performing some further elementary transformations, we obtain

$$\begin{aligned} r_{\omega+\lambda}(a) &= d_P \omega\pi(a) + d_P \lambda\pi(a) + \omega\pi(a^{(1)})\omega\pi(a^{(2)}) + \lambda\pi(a^{(1)})\omega\pi(a^{(2)}) \\ &+ \omega\pi(a^{(1)})\lambda\pi(a^{(2)}) + \lambda\pi(a^{(1)})\lambda\pi(a^{(2)}) = r_\omega(a) + D_\omega \lambda\pi(a) - \lambda\pi(a^{(2)})\omega\pi(\kappa(a^{(1)})a^{(3)}) \\ &+ \lambda\pi(a^{(1)})\omega\pi(a^{(2)}) + \omega\pi(a^{(1)})\lambda\pi(a^{(2)}) + \lambda\pi(a^{(1)})\lambda\pi(a^{(2)}) = r_\omega(a) + D_\omega \lambda\pi(a) \\ &+ \lambda\pi(a^{(1)})\lambda\pi(a^{(2)}) + \omega\pi(a^{(1)})\lambda\pi(a^{(2)}) + \lambda\pi(a^{(3)})\omega[\pi(a^{(1)}) \circ (\kappa(a^{(2)})a^{(4)})] \\ &= r_\omega(a) + D_\omega \lambda\pi(a) + \lambda\pi(a^{(1)})\lambda\pi(a^{(2)}) + \ell_\omega(\pi(a^{(1)}), \lambda\pi(a^{(2)})). \end{aligned}$$

There is the particular context of a quantum principal bundle in which the structure group is interpreted as a ‘discrete object’ and so the algebra of horizontal forms comes equipped with a natural ‘zero curvature’ covariant derivative $D: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$, which is actually an arbitrary differential. In such a context

our starting point is a graded differential $*$ -algebra, called $\mathfrak{h}\mathfrak{o}\mathfrak{t}(P)$ say, equipped with a differential D and quantum group action $F_\wedge: \mathfrak{h}\mathfrak{o}\mathfrak{t}(P) \rightarrow \mathfrak{h}\mathfrak{o}\mathfrak{t}(P) \otimes \mathcal{A}$ so that

$$(A.21) \quad F_\wedge D = (D \otimes \text{id})F_\wedge.$$

In particular, D satisfies a graded Leibniz rule and $D^2 = 0$. In this case, we can construct a natural complete calculus over P as follows. The graded $*$ -algebra is given at the level of vector spaces by

$$(A.22) \quad \Omega(P) = \mathfrak{h}\mathfrak{o}\mathfrak{t}(P) \otimes \Gamma_{inv}^\wedge,$$

while the product and $*$ -structure are given by

$$\begin{aligned} (\varphi \otimes \vartheta)(\psi \otimes \eta) &= (-)^{\partial\psi\partial\vartheta} \varphi\psi^{(0)} \otimes (\vartheta \circ \psi^{(1)})\eta, \\ (\varphi \otimes \vartheta)^* &= (-)^{\partial\varphi\partial\vartheta} \varphi^{(0)*} \otimes (\vartheta^* \circ \varphi^{(1)*}). \end{aligned}$$

There is a canonical differential $d_P: \Omega(P) \rightarrow \Omega(P)$ given by

$$(A.23) \quad d_P(\varphi \otimes \vartheta) = D(\varphi) \otimes \vartheta + (-)^{\partial\varphi} \varphi^{(0)} \otimes \pi(\varphi^{(1)})\vartheta + (-)^{\partial\varphi} \varphi \otimes d^\wedge(\vartheta).$$

Here the differential $d^\wedge: \Gamma_{inv}^\wedge \rightarrow \Gamma_{inv}^\wedge$ is completely determined by its action on quantum germs and is given by

$$(A.24) \quad d^\wedge \pi(a) = -\pi(a^{(1)})\pi(a^{(2)}).$$

This can then be extended to the whole algebra Γ_{inv}^\wedge .

The map $d_P: \Omega(P) \rightarrow \Omega(P)$ defined by the above formula satisfies

$$(A.25) \quad d_P^2 = 0 \quad d_P* = *d_P \quad d_P(\mu\nu) = d_P(\mu)\nu + (-)^{\partial\mu} \mu d_P(\nu)$$

for all $\mu, \nu \in \Omega(P)$.

It is easy to verify that this is indeed a differential calculus on P in the sense of our general definition. In particular there exists a canonical differential extension of the co-action map to $\widehat{F}: \Omega(P) \rightarrow \Omega(P) \widehat{\otimes} \Gamma^\wedge$ which is explicitly given by

$$(A.26) \quad \widehat{F}(\varphi \otimes \Theta) = \varphi^{(0)} \otimes \Theta^{(1)} \otimes \varphi^{(1)} \Theta^{(2)},$$

where $\varphi \in \mathfrak{h}\mathfrak{o}\mathfrak{t}(P)$ and $\Theta \in \Gamma_{inv}^\wedge$. Here we have put $\varphi^{(0)} \otimes \varphi^{(1)} = F_\wedge(\varphi)$ as well as $\Theta^{(1)} \otimes \Theta^{(2)} = \widehat{\phi}(\Theta)$, let us also mention that $\widehat{\phi}(\Gamma_{inv}^\wedge) \subseteq \Gamma_{inv}^\wedge \widehat{\otimes} \Gamma^\wedge$.

Let us also observe that we have a distinguished connection $\varpi: \Gamma_{inv} \rightarrow \Omega^1(P)$ given by

$$(A.27) \quad \varpi(\theta) = 1 \otimes \theta.$$

The covariant derivative of this connection is simply

$$(A.28) \quad D_\varpi(\varphi) = d_P(\varphi) - (-)^{\partial\varphi} \varphi^{(0)} \otimes \pi(\varphi^{(1)}) = D(\varphi).$$

This calculus has been used as a natural framework for Dunkl operators in the main part of this paper, where P is simply an open subset of a finite-dimensional Euclidean space E and G is a Coxeter group acting on this space and associated to a root system R .

APPENDIX B. DIFFERENTIAL CALCULI ON FINITE GROUPS

In this appendix we shall assume that G is any finite group. We are going to describe all bicovariant $*$ -covariant calculi over G .

Since in classical differential geometry a finite group is a zero-dimensional manifold (whose infinitesimal structure is accordingly encoded in its zero-dimensional tangent spaces and so is trivial), it is quite remarkable that in quantum group theory there exist non-trivial differential (that is, ‘infinitesimal’) calculi for any classical finite group. However, there is always more than one such differential calculus (unless the group is trivial).

For the purposes of our paper, the principal interest will be when G is a finite Coxeter group. (See [11] and [12].) However the main formulas here apply for all finite groups.

We give G the discrete topology (i.e., all subsets are open), since this is the only topology on G that is Hausdorff. Also, G is compact with this topology. So this is a simple case of the Gelfand-Naimark theory, where in general one associates to a compact, Hausdorff space its (dual) C^* -algebra of continuous, complex-valued functions. In this case, since every function on G is continuous with respect to the discrete topology, the algebra we get is

$$\mathcal{A} = \{\beta \mid \beta : G \rightarrow \mathbb{C}, \text{ where } \beta \text{ is an arbitrary function}\}.$$

So the algebra \mathcal{A} is commutative (with respect to pointwise multiplication of functions) and finite-dimensional, since its dimension is equal to the number of elements in G .

The elements of the group G in a natural way label a basis in \mathcal{A} by associating to every $g \in G$ the Kronecker delta function whose support is the subset $\{g\}$ of G . We denote this Kronecker delta by g . So for $g, q \in G$ we have

$$(B.1) \quad \sum_{g \in G} g = 1 \quad q \cdot g = \begin{cases} q & \text{when } q = g, \\ 0 & \text{otherwise.} \end{cases}$$

In the above formula, the symbol \cdot is used to denote the product of elements of G , as already described. The coproduct and antipode act on these basis elements by

$$(B.2) \quad \phi(g) = \sum_{h \in G} h \otimes (h^{-1}g), \quad \kappa(g) = g^{-1}.$$

These formulas follow from the general definition that ϕ and κ are the pull-backs to functions (i.e., elements of \mathcal{A}) of the group multiplication $G \times G \rightarrow G ((g, h) \mapsto gh)$ and group inversion $G \rightarrow G (g \mapsto g^{-1})$, respectively. We also have the following expression

$$(B.3) \quad \text{ad}(g) = \sum_{h \in G} (hgh^{-1}) \otimes h$$

for the adjoint action. The elements $g \in G$ also can be interpreted via the map of evaluation at g as characters on \mathcal{A} : For all $\beta \in \mathcal{A}$ we can write $g(\beta) = \beta(g)$. In particular, the counit map $\epsilon : \mathcal{A} \rightarrow \mathbb{C}$, being a character, corresponds to the neutral element $\epsilon \in G$, and so this justifies our using the same symbol ϵ for them.

The ideals in \mathcal{A} are all $*$ -ideals and are naturally labeled by subsets of G , consisting of all points where all the elements of the ideal vanish. In the theory of differential calculi, we are interested in the ideals \mathcal{R} satisfying $\mathcal{R} \subseteq \ker(\epsilon)$. Such

ideals correspond to subsets of G containing the neutral element ϵ . Hence there is a natural correspondence between these ideals \mathcal{R} and subsets S of $G \setminus \{\epsilon\}$. We can write $\mathcal{R} = \mathcal{R}_S$, for such ideals \mathcal{R} and $S = S_{\mathcal{R}}$ for such subsets. To put it in terms of precise expressions we have

$$\mathcal{R}_S = \left\{ f \in \ker(\epsilon) \mid f(x) = 0 \quad \forall x \in S \right\}, \quad S_{\mathcal{R}} = \left\{ x \in G \setminus \{\epsilon\} \mid f(x) = 0 \quad \forall f \in \mathcal{R}_S \right\}.$$

Because of the natural isomorphism $\Gamma_{inv} \leftrightarrow \ker(\epsilon)/R$, as explained in [23] and [7], there is a canonical basis in the space Γ_{inv} given by the elements $\{\pi(g) = [g] \mid g \in S_{\mathcal{R}}\}$. The right \mathcal{A} -module structure \circ is specified by

$$(B.4) \quad [g] \circ h = \begin{cases} [g] & \text{if } g = h, \\ 0 & \text{otherwise,} \end{cases}$$

and the complete right \mathcal{A} -module structure on the calculus Γ is defined by

$$(B.5) \quad [g]h = (hg^{-1})[g].$$

It is worth mentioning that

$$(B.6) \quad [\epsilon] = - \sum_{s \in S} [s],$$

as follows from the fact that elements $g \in G$ sum up to 1, and that the germs map π acts non-trivially on S and ϵ only. The property of $*$ -covariance $\kappa(\mathcal{R})^* = \mathcal{R}$ is equivalent to $S_{\mathcal{R}}^{-1} = S_{\mathcal{R}}$ or, in other words, the set S should contain with every element s its inverse s^{-1} .

On the other hand, the bicovariance property $\text{ad}(\mathcal{R}) \subseteq \mathcal{R} \otimes \mathcal{A}$ is equivalent to $gS_{\mathcal{R}}g^{-1} = S_{\mathcal{R}}$ for all $g \in G$ or, in other words, the set $S_{\mathcal{R}}$ is invariant under the adjoint action of G . Equivalently $S_{\mathcal{R}}$ can be written as a (disjoint) union of conjugation classes.

In particular, we see that irreducible $*$ -covariant bicovariant calculi are in a natural correspondence with the subsets of G which are either (i) a single conjugate class that coincides with its inverse or (ii) a union of two distinct mutually inverse conjugation classes. The notion of irreducibility is understood as the non-existence of another non-trivial differential $*$ -covariant and bicovariant calculus onto which the given calculus can be projected.

The canonical braid-operator $\sigma: \Gamma_{inv}^{\otimes 2} \rightarrow \Gamma_{inv}^{\otimes 2}$ of [23] can be calculated to be

$$(B.7) \quad \sigma([h] \otimes [g]) = [hgh^{-1}] \otimes [h].$$

We have in particular that if $gh = hg$, then σ acts as the standard flip. The elements $[g] \otimes [g]$ are always σ -symmetric. In general, this operator will have eigenvalues which are complex roots of unity, since the above formula gives us a bijective map from the finite set $S \times S$ to itself, in other words

$$\sigma: (h, g) \mapsto (hgh^{-1}, g).$$

So the orbits of this action are used to construct the eigenvectors of σ . Note that the diagonal elements $[g] \times [g]$ are the only one-element orbits, and the two-element orbits are precisely given by $\{(h, g), (g, h)\}$ where g and h are different elements of $S_{\mathcal{R}}$ that commute. Each orbit naturally generates the space of a regular representation of a finite cyclic group whose order is equal to the number of elements in that orbit.

In terms of all these identifications, the universal differential envelope Γ^\wedge of Γ is constructible in terms of the following quadratic relations:

$$(B.8) \quad \sum_{gq=h} [g] \otimes [q] \quad \epsilon \neq h \notin S_{\mathcal{R}} \quad q, g \in S_{\mathcal{R}}.$$

In particular, we see that the dimension of the quadratic relations space in $\Gamma_{inv} \otimes \Gamma_{inv}$ is the same as the cardinality of the set of non-identity elements of $G \setminus S_{\mathcal{R}}$ which can be expressed as a product of two elements from $S_{\mathcal{R}}$.

Let us now apply all this in the context of quantum principal bundles and maps $T_\lambda: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)$ induced by connection displacements λ . Such maps naturally appear in the expressions for covariant derivatives and are given by

$$(B.9) \quad T_\lambda(\varphi) = \varphi^{(0)} \lambda \pi(\varphi^{(1)}).$$

Taking into account the structural form of the calculus and in particular the identity (B.6) we obtain

$$(B.10) \quad T_\lambda(\varphi) = \sum_{s \in S} (\varphi_s - \varphi) \lambda[s],$$

where we used an alternative form to describe the action F_\wedge via

$$(B.11) \quad F_\wedge(\varphi) = \varphi^{(0)} \otimes \varphi^{(1)} = \sum_{g \in G} \varphi_g \otimes g$$

so that

$$\varphi_g = (\text{id} \otimes g) F_\wedge(\varphi).$$

Here $g \in G$ is interpreted as a character $g: \mathcal{A} \rightarrow \mathbb{C}$ via evaluation as described earlier.

The displaced covariant derivative is thus given by

$$(B.12) \quad D_{\omega+\lambda}(\varphi) = D_\omega(\varphi) + (-)^{\partial\varphi} \sum_{s \in S} (\varphi - \varphi_s) \lambda[s].$$

The displacement map λ is $*$ -invariant, which translates into

$$(B.13) \quad (\lambda[s])^* = -\lambda[s^{-1}].$$

It is also ad-covariant, which means that

$$F_\wedge \lambda[s] = \sum_{g \in G} (\lambda[s])_g \otimes g = \lambda[s^{(2)}] \otimes \kappa(s^{(1)}) s^{(3)} = \sum_{g \in G} \lambda[gs g^{-1}] \otimes g.$$

This can be simply expressed as

$$(B.14) \quad \lambda[s]_g = \lambda[gs g^{-1}]$$

for every $s \in S$ and $g \in G$.

Now for an irreducible, bicovariant and $*$ -covariant calculus Γ these properties tell us that the displacement λ is completely determined by the value $\lambda[s] \in \mathfrak{hor}^1(P)$ on a single element $[s]$ where $s \in S$. In the case when the calculus is based on the union of two disjoint conjugation classes (each being the inverse of the other), we can take $\lambda[s]$ to be an arbitrary horizontal one-form and then use (B.13) and (B.14) to define λ on all of Γ_{inv} .

On the other hand, when there is only one conjugation class (being its own inverse) used to define the calculus, we have

$$\lambda[s]^* = -\lambda[s]_q,$$

where $q \in G$ is such that $s^{-1} = qsq^{-1}$. The same applies in a more general context, for arbitrary elements of S conjugated to their inverse.

In the general case of an arbitrary bicovariant and $*$ -covariant calculus Γ the above reasoning should be repeated for all the irreducible blocks.

Let us now consider an important special case of the above situation, when the algebra $\mathfrak{hor}(P)$ admits a ‘local trivialization’. This means that we can establish in a covariant way projection maps of the form

$$(B.15) \quad \mathfrak{hor}(P) \rightsquigarrow \mathcal{L} \otimes \mathcal{A},$$

where \mathcal{L} is a ‘local representation’ of the calculus on the base space M . For example, we can consider a completely classical horizontal forms algebra, where M and P are classical, and $\mathfrak{hor}(P)$ is the classical algebra of differential forms on P .

Then locally the map λ is given by

$$(B.16) \quad \lambda[s] \leftrightarrow Y[s^{(2)}] \otimes \kappa(s^{(1)})s^{(3)} = \sum_{g \in G} Y[gs g^{-1}] \otimes g,$$

where $Y: \Gamma_{inv} \rightarrow \mathcal{L}$ is a linear map. In this form the covariance condition is automatically satisfied. The $*$ -condition reflects as an appropriate restriction on Y . For example, if the local trivialization intertwines the $*$ -structure on $\mathfrak{hor}(P)$ and the product of $*$ -structures on \mathcal{L} and \mathcal{A} , then

$$(B.17) \quad (Y[s])^* = -Y[s^{-1}].$$

We have covered the main properties of the differential calculi and connections associated to finite groups G , interpreted as structure groups of quantum principal bundles. For the purposes of this paper we are interested in the special case when G is a Coxeter group. Specific formulas and considerations for that case are collected in Section 3.

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REFERENCES

- [1] F. Chouchene, L. Gallardo and M. Mili, *Les équations de la chaleur et de Poisson pour le laplacien généralisé de Jacobi-Dunkl*, C.R. Acad. Sci. Paris, Ser. I **341** (2005) 179–184.
- [2] A. Connes, Non-commutative differential geometry, I.H.E.S., Ext. Publ. Math. **62**, 1986.
- [3] V. Drinfeld, Quantum groups, Proc. Int. Cong. Math., Berkeley (1986) 793–820.
- [4] C. Dunkl, Differential-difference operators associated to reflection groups, Trans. Am. Math. Soc. **311** (1989) 167–183.
- [5] C. Dunkl and E. Opdam, Dunkl operators for complex reflection groups, Proc. London Math. Soc. (3) **86** (2003), 70–108.
- [6] C. Dunkl and Y. Xu, *Orthogonal Polynomials of Several Variables*, Cambridge Univ. Press, 2001.

- [7] M. Đurđevich, Geometry of Quantum Principal Bundles I, *Commun. Math. Phys.* **175** (3) (1996) 457–521.
- [8] M. Đurđevich, Geometry of Quantum Principal Bundles II – Extended Version, *Rev. Math. Phys.* **9** (5) (1997) 531–607.
- [9] M. Đurđevich, Geometry of Quantum Principal Bundles III – Structure of Calculi and Around, *Algebras, Groups and Geometries*, to appear (2011).
- [10] M. Đurđevich, Characteristic Classes of Quantum Principal Bundles – Extended Version, *Algebras, Groups and Geometries* **26** (3) (2009) 241–341.
- [11] L.C. Grove and C.T. Benson, *Finite Reflection Groups* Springer, 1985.
- [12] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.
- [13] M.F.E. de Jeu, The Dunkl transform, *Invent. Math.* **113** (1993) 147–162.
- [14] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience Publishers, New York, London (1963).
- [15] E. Prugovečki: *Quantum Geometry—A Framework for Quantum General Relativity*, Kluwer Academic Publishers, 1992.
- [16] M. Rösler, Generalized Hermite Polynomials and the Heat Equation for Dunkl Operators, *Commun. Math. Phys.* **192**, (1998) 519–542.
- [17] M. Rösler, Dunkl operators: Theory and Applications, in: *Lecture Notes in Mathematics*, vol. 1817, Eds. E. Koelink and W. van Assche, pp. 93–135, Springer, 2003.
- [18] M. Rösler and M. Voit, Markov Processes Related with Dunkl Operators, *Adv. Appl. Math.* **21** (1998), 577–643.
- [19] E.M. Opdam, *Lectures on Dunkl operators for real and complex reflection groups*, *MSJ Memoirs* **8**, Math. Soc. of Japan, 2000.
- [20] J-P. Serre, *Lie Algebras and Lie Groups*, Benjamin, 1965.
- [21] S.B. Sontz, On Segal-Bargmann analysis for finite Coxeter groups and its heat kernel, accepted for publication in *Mathematische Zeitschrift*. Printed online 14 April 2010.
- [22] S.L. Woronowicz, Compact Matrix Pseudogroups, *Commun. Math. Phys.* **111** (1987) 613–665.
- [23] S.L. Woronowicz, Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups), *Commun. Math. Phys.* **122** (1989) 125–170.

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